

RELATIONS FOR GROTHENDIECK GROUPS OF GORENSTEIN RINGS

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ABSTRACT. We consider the converse of the Butler, Auslander-Reiten's Theorem which is on the relations for Grothendieck groups. We show that a Gorenstein ring is of finite representation type if the Auslander-Reiten sequences generate the relations for Grothendieck groups. This gives an affirmative answer of the conjecture due to Auslander.

1. INTRODUCTION

Throughout this section, (R, \mathfrak{m}) denote a commutative Cohen-Macaulay complete ring with the residue field k . All R -modules are assumed to be finitely generated. We say that an R -module M is Cohen-Macaulay if

$$\mathrm{Ext}_R^i(k, M) = 0 \quad \text{for all } i < \dim R.$$

We denote by $\mathrm{mod}(R)$ the category of R -modules with R -homomorphisms and by \mathcal{C} the full subcategory of $\mathrm{mod}(R)$ consisting of all Cohen-Macaulay R -modules.

Let $K_0(\mathcal{C})$ be a Grothendieck group of \mathcal{C} . Since $K_0(\mathcal{C}) = K_0(\mathrm{mod}(R))$, it is important to investigate $K_0(\mathcal{C})$ for the study of K-theory of $\mathrm{mod}(R)$. Set $G(\mathcal{C}) = \bigoplus_{X \in \mathrm{ind} \mathcal{C}} \mathbb{Z} \cdot [X]$, which is a free abelian group generated by isomorphism classes of indecomposable objects in \mathcal{C} . We denote by $\mathrm{EX}(\mathcal{C})$ a subgroup of $G(\mathcal{C})$ generated by

$$\{[X] + [Z] - [Y] \mid \text{there is an exact sequence } 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \mathcal{C}\}.$$

We also denote by $\mathrm{AR}(\mathcal{C})$ a subgroup of $G(\mathcal{C})$ generated by

$$\{[X] + [Z] - [Y] \mid \text{there is an AR sequence } 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \mathcal{C}\}.$$

On the relation for Grothendieck groups, Butler[3], Auslander-Reiten[2], and Yoshino[8] prove the following theorem.

Theorem 1.1. [3, 2, 8] *If R is of finite representation type then $\mathrm{EX}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$.*

Here we say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay R -modules.

In this note we consider the converse of Theorem 1.1. Actually we shall show the following theorem.

Theorem 1.2. *Let R be a complete Gorenstein local ring with an isolated singularity and with algebraically closed residue field. If $\mathrm{EX}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$, then R is of finite representation type.*

Date: April 25, 2016.

2000 Mathematics Subject Classification. Primary 13D15 ; Secondary 16G50, 16G60.

Key words and phrases. Grothendieck group, finite representation type, AR sequence.

This work was partly supported by JSPS Grant-in-Aid for Young Scientists (B) 15K17527.

Auslander conjectured the converse of Theorem 1.1 is true. It has been proved by Auslander[1] for Artin algebras and by Auslander-Reiten[2] for complete one dimensional domain. Our theorem gives an affirmative answer to his conjecture for the case of complete Gorenstein local rings with an isolated singularity.

2. PROOF OF THEOREM 1.2

In the rest of the note, we always assume that (R, \mathfrak{m}) is a complete Gorenstein local ring with the residue field k , where k is an algebraically closed field. For the category of Cohen-Macaulay R -modules \mathcal{C} , we denote by $\underline{\mathcal{C}}$ the stable category of \mathcal{C} . The objects of $\underline{\mathcal{C}}$ are the same as those of \mathcal{C} , and the morphisms of $\underline{\mathcal{C}}$ are elements of $\underline{\text{Hom}}_R(M, N) = \text{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathcal{C}}$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R -modules. Since R is complete, \mathcal{C} , hence $\underline{\mathcal{C}}$, is a Krull-Schmidt category. For $M \in \mathcal{C}$ we denote it by \underline{M} to indicate that it is an object of $\underline{\mathcal{C}}$. For a finitely generated R -module M , take a free presentation

$$\cdots \rightarrow F_1 \xrightarrow{d} F_0 \rightarrow M \rightarrow 0.$$

We denote $\text{Im } d$ by ΩM , which is called a (first) syzygy of M . And we also denote by $\text{tr} M$ the cokernel $F_0^* \xrightarrow{d^*} F_1^*$ where $(-)^* = \text{Hom}_R(-, R)$.

First of all we prepare a key lemma.

Lemma 2.1. *There exists $X \in \mathcal{C}$ such that $\underline{\text{Hom}}_R(M, X) \neq 0$ for all $M \in \mathcal{C}$ with $\underline{M} \neq \underline{0}$ in $\underline{\mathcal{C}}$.*

Proof. Take a Cohen-Macaulay approximation of the residue field k as X ;

$$(2.1) \quad 0 \rightarrow Y \rightarrow X \rightarrow k \rightarrow 0.$$

Then we shall show X satisfies the assertion of the lemma.

Let M be a non free Cohen-Macaulay module, that is, $\underline{M} \neq \underline{0}$ in $\underline{\mathcal{C}}$. Apply $\text{Hom}_R(M, -)$ to the sequence (2.1), we have the commutative diagram with exact rows, where the vertical arrows are natural surjections;

$$\begin{array}{ccccccc} \underline{\text{Hom}}_R(M, X) & \longrightarrow & \underline{\text{Hom}}_R(M, k) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \text{Hom}_R(M, Y) & \longrightarrow & \text{Hom}_R(M, X) & \longrightarrow & \text{Hom}_R(M, k) \longrightarrow 0. \end{array}$$

Assuming $\underline{\text{Hom}}_R(M, X) = 0$, we have $\underline{\text{Hom}}_R(M, k) = 0$ from this diagram. On the other hand, since $\underline{\text{Hom}}_R(M, k) = \text{Tor}_R^1(\text{tr} M, k)$ ([8, Lemma 3.9]), this implies that $\text{tr} M$ is free ([6, §19 Lemma 1(i)]), so that M is free. This is a contradiction. \square

The stable category $\underline{\mathcal{C}}$ has a structure of a triangulated category since R is Gorenstein (cf. [4]). By the definition of a triangle, $\underline{L} \rightarrow \underline{M} \rightarrow \underline{N} \rightarrow \underline{L}[1]$ is a triangle in $\underline{\mathcal{C}}$ if and only if there is an exact sequence $0 \rightarrow L \rightarrow M' \rightarrow N \rightarrow 0$ in \mathcal{C} with $\underline{M'} \cong \underline{M}$ in $\underline{\mathcal{C}}$. To prove our theorem, we use a theory of Auslander-Reiten (abbr. AR) triangles. The notion of AR triangles is a stable analogy of AR sequences.

Definition 2.2. [4, Chapter I, §4] We say that a triangle $\underline{Z} \rightarrow \underline{Y} \xrightarrow{f} \underline{X} \xrightarrow{w} \underline{Z}[1]$ in $\underline{\mathcal{C}}$ is an AR triangle ending in \underline{X} (or starting from \underline{Z}) if it satisfies

- (1) \underline{X} and \underline{Z} are indecomposable.

- (2) $\underline{w} \neq 0$.
 (3) If $\underline{g} : \underline{W} \rightarrow \underline{X}$ is not a split epimorphism, then there exists $\underline{h} : \underline{W} \rightarrow \underline{Y}$ such that $\underline{g} = \underline{f} \circ \underline{h}$.

Remark 2.3. Let $0 \rightarrow Z \rightarrow Y \xrightarrow{f} X \rightarrow 0$ be an AR sequence in \mathcal{C} . Then $\underline{Z} \rightarrow \underline{Y} \xrightarrow{f} \underline{X} \rightarrow \underline{Z}[1]$ is an AR triangle in $\underline{\mathcal{C}}$. See [5, Proposition 2.2] for example.

We say that (R, \mathfrak{m}) is an isolated singularity if each localization $R_{\mathfrak{p}}$ is regular for each prime ideal \mathfrak{p} with $\mathfrak{p} \neq \mathfrak{m}$. Note that if R is an isolated singularity, \mathcal{C} admits AR sequences (cf. [8, Theorem 3.2]). Hence $\underline{\mathcal{C}}$ admits AR triangles (Remark 2.3). We also note that since we have the isomorphism $\underline{\text{Hom}}_R(M, N) \cong \text{Tor}_1^R(\text{tr} M, N)$ for finitely generated R -modules M and N , one can show that $\text{length}_R(\underline{\text{Hom}}_R(M, N))$ is finite for $M, N \in \underline{\mathcal{C}}$ if R is an isolated singularity. When U is indecomposable in \mathcal{C} then denote by $\mu(\underline{U}, \underline{X})$ the multiplicity of \underline{U} as a direct summand of \underline{X} .

Proposition 2.4. [7, Proposition 3.1][5, Proposition 2.14 (1)] *Let R be an isolated singularity and let $\underline{Z} \xrightarrow{g} \underline{Y} \xrightarrow{f} \underline{X} \rightarrow \underline{Z}[1]$ be an AR triangle in $\underline{\mathcal{C}}$. Then the following equality holds for each indecomposable $U \in \mathcal{C}$:*

$$[\underline{U}, \underline{X}] + [\underline{U}, \underline{Z}] - [\underline{U}, \underline{Y}] = \mu(\underline{U}, \underline{X}) + \mu(\underline{U}, \underline{X}[-1]).$$

Here $[\underline{U}, \underline{X}]$ is an abbreviation of $\text{length}_R(\underline{\text{Hom}}_R(M, N))$.

Proof of Theorem 1.2. Let X be the module that satisfies the conditions as in Lemma 2.1. Take the syzygy of X .

$$0 \rightarrow \Omega X \rightarrow P \rightarrow X \rightarrow 0.$$

By the assumption, since $\text{EX}(\mathcal{C}) = \text{AR}(\mathcal{C})$, we have the equality in $\text{G}(\mathcal{C})$,

$$[X] + [\Omega X] - [P] = \sum_{M \in \text{ind} \mathcal{C}}^{\text{finite}} a_{M, X}([M] + [\tau M] - [E_M]),$$

where $[M] + [\tau M] - [E_M]$ come from AR sequences $0 \rightarrow \tau M \rightarrow E_M \rightarrow M \rightarrow 0$. The equality yields that

$$(2.2) \quad [\underline{U}, \underline{X} \oplus \underline{\Omega X}] = \sum_{M \in \text{ind} \mathcal{C}}^{\text{finite}} a_{M, X}([\underline{U}, \underline{M}] + [\underline{U}, \underline{\tau M}] - [\underline{U}, \underline{E_M}])$$

for each $U \in \mathcal{C}$. Since $\underline{\tau M} \rightarrow \underline{E_M} \rightarrow \underline{M} \rightarrow \underline{\tau M}[1]$ are AR triangles (Remark 2.3), by Proposition 2.4, we see that there are only a finite number of indecomposable modules in \mathcal{C} that makes the RHS in (2.2) non-zero, so is LHS. By Lemma 2.1, we conclude that $\underline{\mathcal{C}}$, hence \mathcal{C} is of finite representation type. \square

ACKNOWLEDGMENTS

The author express his deepest gratitude to Osamu Iyama and Yuji Yoshino for valuable discussions and helpful comments. The author also thank the referee for his/her careful reading.

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